Dynamic Pricing Strategies with Reference Effects

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We consider the dynamic pricing problem of a monopolist firm in a market with repeated interactions, where demand is sensitive to the firm’s pricing history. Consumers have memory and are prone to human decision making biases and cognitive limitations. As the firm manipulates prices, consumers form a reference price that adjusts as an anchoring standard based on price perceptions. Purchase decisions are made by assessing prices as discounts or surcharges relative to the reference price, in the spirit of prospect theory.

We prove that optimal pricing policies induce a perception of monotonic prices, whereby consumers always perceive a discount, respectively surcharge, relative to their expectations. The effect is that of a skimming or penetration strategy. The firm’s optimal pricing path is monotonic on the long run, but not necessarily at the introductory stage. If consumers are loss averse, we show that optimal prices converge to a constant steady state price, characterized by a simple implicit equation; otherwise the optimal policy cycles. The range of steady states is wider the more loss averse consumers are. Steady state prices decrease with the strength of the reference effect, and with customers’ memory, all else equal. Offering lower prices to frequent customers may be suboptimal, however, if these are less sensitive to price changes than occasional buyers.

If managers ignore such long term implications of their pricing strategy, the model indicates that they will systematically price too low and lose revenue. Our results hold under very general reference dependent demand models.

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1. Introduction

Traditional economic, marketing and operational models view the consumer as a rational agent who makes decisions based on current prices, income and market conditions. In a market with repeated interactions, such as frequently purchased consumer goods (e.g. gasoline), services (e.g. resort hotels, individual insurance) and B2B settings (e.g. media broadcasting, industrial maintenance), customers’ purchase decisions are also determined by past observed prices.

As customers revisit the firm, they develop price expectations, or reference prices, which become
the benchmark against which current prices are compared. Prices above the reference price appear to be “high”, whereas prices below the reference price are perceived as “low”. The latter effect stimulates short term demand and provides incentives for retailers to run price promotions as a mechanism to increase short-term profits. On the other hand, price promotions decrease consumers’ price expectations, and hence their willingness to buy the product at higher prices in the future. (We ignore stock-piling effects and assume that consumers are fully informed about product quality, and do not judge quality level by price.) For the firm, this means that high profits today may come at the expense of a loss in future demand, and hence less profit in the future. Therefore, a profit maximizing firm must consider the long term implications of its pricing strategy.

Our goal is to characterize what types of pricing strategies are optimal in such repeated interaction markets. We investigate when the firm should use traditional skimming or penetration strategies, and whether a constant price versus a cycling policy is optimal in the long run. A distinctive feature of this work, compared to traditional microeconomic and operational frameworks, is that it relies on descriptive models of consumer behavior to derive pricing prescriptions under complex dynamics. Ultimately, we aim to provide normative insights on how behavioral demand parameters, such as customer loyalty and loss aversion, should reshape and reflect in managerial pricing decisions when customers have repetitive, long term relationships with the firm.

The marketing literature provides compelling empirical evidence for the dependence of demand on past prices. Adaptation level theory (Helson 1964) predicts that customers respond to the current price of a product by comparing it to an internal standard formed based on past price exposures, called the reference price. While other reference price models exist in the literature, an empirical comparison conducted by Briesch et al. (1997) shows that “the best [...] model is [...] one that is based on the brand’s own price history”, i.e. an internal reference price mechanism.

The impact of the reference price on demand, called reference effect, is behaviorally explained at the individual level by the postulates of prospect theory (Kahneman and Tversky 1979). Accordingly, consumers perceive prices as gains (discounts) or losses (surcharges) relatively to a reference price, and there is an inherent asymmetry in perception, in that losses loom larger than gains of the same magnitude (loss aversion). In addition, there is diminishing sensitivity to both gains and losses. The vast empirical validation of prospect theory in the context of reference prices is best synthesized by Kalyanaram and Winer (1995). Their “key empirical generalizations” validate the kinked S-shaped reference effect at the aggregate demand level. The value of this theory to managers and researchers alike is that “it predicts how consumers actually behave, rather than how they ought to behave” (Nagle and Holden 1995).
With few exceptions, the dynamic pricing literature is oblivious of such behavioral aspects underlying demand. Recent surveys on dynamic pricing confirm that the state of the art models unrealistically assume demand to be given exogenously, and customers’ purchase decisions to be based solely on the current price posted by the seller (Bitran and Caldentey 2003, Elmaghraby and Keskinoçak 2003). Similarly, in industry practice, prices are typically based on empirically estimated demand models that reflect consumer response conditional on current prices only, indicating that firms typically follow myopic pricing policies.

Recent research trends in dynamic pricing and revenue management investigate the issue of strategic buyer behavior, whereby consumers optimally time their purchase in anticipation of future prices (Ovchinnikov and Milner 2005, Liu and van Ryzin 2005). In parallel work, Heidhues and Köszegi (2005) propose an economic model where consumers’ stochastic reference points (for consumption and price) are determined as a rational expectation “personal” equilibrium. They investigate conditions for price stickiness in a static monopolistic framework with random cost.

In contrast, we are interested in incorporating the impact of past prices on demand in a dynamic setting. Our consumers are more realistic than the traditional “homo economicus”, yet boundedly rational. They have memory and are prone to human decision making biases (such as anchoring effects and loss aversion) and cognitive limitations. In particular, they are unaware of their biases and do not act strategically.

There are few demand models similar to ours in the dynamic pricing literature. Sorger (1988) studies local stability of joint dynamic pricing and advertising policies under reference price effects, but his results are not comparable to ours due to the compound advertising effect. Also, his Assumption 2 c. is inconsistent with our model.

Kopalle et al. (1996) and Fibich et al. (2003) – henceforth KRA, respectively FGL – investigate a problem similar to ours under a much simpler linear demand model with (kinked) linear reference effects. KRA provide computational insights from a discrete time formulation (adapted from Greenleaf 1995), but their analytical results are limited. They conjecture and verify numerically (on p.66) the monotonic convergence of optimal prices under asymmetric demand effects; some special cases are proved in their Proposition 2. An elegant proof of this conjecture is due to FGL in a continuous time control framework. Their approach, contingent on a linear demand model, reduces the problem to solving a linear Euler equation. This cannot be solved explicitly for non-linear models, hence “it is natural to ask whether the results of [their] study would remain valid for more general demand functions” (p.729), and precisely what general structural properties, if not
just linearity, drive these results. We extend their insights, and provide stronger analytical results, under very general modeling assumptions on demand.

We provide structural, as opposed to parametric results, that allow to solve the monopolist’s dynamic pricing problem under a most general reference dependent demand model, that captures non-linearities and dynamics in response to changes in the reference price. The model is based on behavioral claims supported by empirical theories; it extends and generalizes existing models used in the literature (Sorger 1988, Greenleaf 1995, KRA, FGL). Our main technical contributions, detailed below, include a simple characterization of the steady state price, as well as convergence and monotonicity properties of the optimal pricing strategy. From a methodological standpoint, our approach suggests new ways of proving structural properties for dynamic programs with kinked rewards, which are the minimum of two smooth functions.

We characterize the existence and uniqueness of an optimal constant pricing strategy on the long run (steady state). If consumers are loss seeking, i.e. more responsive to discounts than surcharges, the optimal policy cycles. If consumers are loss averse there is a range of steady states, reduced to a unique one in the loss neutral case. We provide a simple closed form characterization of the steady state, which allows for insightful sensitivity analysis with respect to behavioral parameters.

The value of the steady state price decreases with customers’ loyalty (memory effects) and with their sensitivity to past prices (reference effects), all else equal. Also, the range of steady state prices is wider the more loss averse consumers are. If customers are heterogeneous in terms of their shopping frequency (and nothing else), retailers should offer discounts to loyal customers, and higher prices to occasional buyers (relatively to the unique price charged in a non-segmented market). While such strategies are consistent with current practices by retailers in consumer goods industries, our results indicate that they are not necessarily optimal if occasional buyers are more responsive to price changes than loyal ones.

The steady state price is bounded between the profit maximizing price charged to consumers who don’t form reference effects, and the steady state price charged by a myopic firm to reference sensitive consumers. We show that firms who ignore the long term impact of reference price effects, will myopically but systematically underprice.

Our results also characterize the transient monotonicity and convergence of price and reference price strategies. In general, we show that the strategic monopolist seeks a monotonic reference price strategy. If consumers’ reference price is initially high, they will be lead to perceive a gain in each period, as the firm will consistently price below the reference price. This perception of monotonic prices has the effect of a skimming strategy. Similarly, a low reference price leads to a penetration
type strategy. While high-low prices may be observed at the introductory stage, on the long run prices are eventually monotonic in the same direction as reference prices. We provide conditions for a monotonic pricing policy to be optimal (i.e. traditional skimming or penetration). This is true for example if consumers’ memory is short (i.e. the reference price equals the last price observed), or if reference prices have increasing marginal impact on demand. In particular, this includes the linear models studied by KRA and FGL.

In summary, our results for general reference dependent demand models confirm the robustness of FGL and KRA insights concerning stability and steady state analysis, but not necessarily monotonicity of the transient pricing policy. The latter appears to be a result of their linear reference effect assumption. Our approach is similar in spirit to the two-stage method in FGL, but our analysis relies on structural arguments, as opposed to explicit solutions of a parametric model.

The remaining of the paper is structured as follows. Section 2 describes a very general model of reference dependent demand based on behavioral findings, in particular prospect theory. We summarize behavioral claims that form the base of our various, parsimonious assumptions in the rest of the paper. Section 3 presents the dynamic pricing model. The next two sections are structured around consumer’s loss aversion. Section 4 focuses on models where consumers are loss neutral, with the loss seeking case briefly addressed at the end. We investigate the existence of the steady state, and stability and monotonicity properties of the optimal price and reference price strategies; these are benchmarked against myopic policies. Section 5 studies the same issues when consumers are loss averse, i.e. the reference effect is consistent with prospect theory. Section 6 concludes our findings.

2. General Reference Dependent Demand Model

This section describes how consumers make purchase decisions based on prices and reference prices. The most prominent theory of reference dependent preferences is prospect theory (Kahneman and Tversky 1979); the deterministic versions in Thaler (1985) and Tversky and Kahneman (1991) are most relevant to our context.

The mental accounting framework (Thaler 1985) proposes that the total utility from purchasing a product consists of two components: acquisition utility and transaction utility. The former corresponds to the monetary value of the deal, determined by the discrepancy between price and the value of the product to the consumer. Transaction utility corresponds to the psychological value of the deal, determined by the discrepancy between price and reference price. Thus, reference prices affect customer behavior via the magnitude of the perceived “gain” or “loss” \( x = r - p \) relative to
the reference point $r$. The microeconomic demand model induced by such individual preferences is derived by Putler (1992), who validates it empirically with egg sales data. Empirical validation of prospect theory at the demand level is remarkably synthesized by Kalyanaram and Winer (1995).

Define the reference effect on demand $R(r - p, r) = D(p, r) - D(p, p)$ as the differential demand at price $p$ due to the reference price being $r$ instead of the actual price $p$. Alternatively, $R(x, r)$ measures the impact on demand of a perceived discount/surcharge $x = r - p$ relative to the reference price $r$. Empirical evidence suggests that the larger magnitude of the perceived gain/loss, the larger the corresponding absolute impact on demand; this is known as Reference Dependence (RD).

Furthermore, it has been observed that for normal goods purchase experience with higher past prices increases the likelihood of buying the product for a given observed price. This leads to the following standing assumption on the demand model:

**Assumption 1 (Reference Dependent Demand).** Demand $D(p, r)$ is non-negative bounded and continuous, decreasing in price $p$ and increasing in the reference price $r$. The corresponding reference effect $R(x, r)$ is increasing in $x$.

In particular, the definition of the reference effect together with monotonicity of demand implies:

**Remark 1.** $R(x, r) \geq 0$ for $x > 0$, $R(x, r) \leq 0$ for $x < 0$ and $R(0, r) = 0$. Furthermore, $R(r - p, r)$ is increasing in $r$.

We further describe the reference dependent demand model in terms of (1) how the perceived discount/surcharge $x$ impacts demand and (2) how this impact varies with shifts in the reference value $r$. We provide behavioral claims, based on empirical generalizations and established theories of consumer behavior. These claims are the base of our parsimonious modeling assumptions in the rest of the paper.

### 2.1. Gain-Loss Asymmetries

Behavioral theories suggest that consumers may react differently to discounts vs. surcharges. In particular, surcharges loom larger than discounts of the same magnitude (loss aversion). To measure these effects, we use the index of loss aversion $\rho(r) = \frac{\lambda_-(r)}{\lambda_+(r)}$, where $\lambda_-(r), \lambda_+(r)$ denote the left and right $x$-derivatives of $R(x, r)$ at $x = 0$ (Köbberling and Wakker 2005). People are loss averse at a given reference point $r$ whenever $\rho(r)$ exceeds 1. Asymmetries are considered robust, i.e. in a particular context consumers are unlikely to be loss averse for certain reference levels and loss seeking for others. This leads to the following alternative claims:

- **Loss Aversion (LA):** $\rho(\cdot) > 1$;
- **Loss Neutrality (LN):** $\rho \equiv 1$;
- **Loss Seeking (LS):** $\rho(\cdot) < 1$. 
There is strong empirical evidence on loss aversion in the pricing context, with an observed index of loss aversion between 1.7-2.5 (see e.g. Ho and Zhang 2005, and references therein). In addition to reference dependence and loss aversion, prospect theory postulates that consumers exhibit diminishing sensitivity to both discounts and surcharges, leading to a kinked S-shaped reference effect.

**Diminishing Sensitivity (DS):** \( R(x, r) \) is concave for \( x \geq 0 \) and convex for \( x \leq 0 \) for all \( r \).

**Prospect Theory (PT):** (RD), (LA) and (DS) hold for \( R(x, r) \).

The pricing implications of prospect theory are further investigated in Section 5. Models with loss neutral and loss seeking buyers are investigated in Section 4.

### 2.2. Shifting Reference Price

Limited work has been done to characterize dynamics of the reference effect when the status quo varies. A distinctive feature of our work is to incorporate such effects in the most general manner.

Kahneman and Tversky (1979, p.277) suggest that the value function becomes more flat as the reference point increases. This is supported by empirical evidence from mental accounting studies (e.g. Thaler 1999). For example, many people would be happy to spend half an hour on the internet for a 50 Euro discount on a 100 Euro airplane ticket, but not on a 2000 Euro one. In both cases, however, costs (30 minutes) and benefits (50 Euro) are identical, so from a purely rational view point the trade-off is the same. This example indicates that consumers are less responsive to a given price change when the status quo price is higher. The following claim, adopted from the axiomatization of deference dependence by Schmidt (2003), captures this effect:

**Decreasing Curvature (DC):** The marginal impact of a gain/loss \( x \) on demand decreases with the reference value \( r \).

Recall that a real-valued function \( f(x, y) \) is supermodular if it has increasing differences, i.e. \( f(x_1, y) - f(x_2, y) \) is increasing in \( y \) for all \( x_1 > x_2 \). If \( f \) is differentiable, this amounts to positivity of the cross-partial derivative \( f_{xy}(x, y) \geq 0 \) (partial derivatives are denoted by corresponding subscripts). Submodularity is defined by the opposite inequality.

Claim (DC) states that \( R(x, r) \) is submodular on the gain, respectively loss domain; this is irrespective of the gain-loss asymmetries discussed previously. An illustration of this claim under prospect theory is provided in Figure 1.

### 2.3. Absolute and Relative Difference Models.

We next provide simple examples of reference effects used in the literature. The most common is the absolute difference (AD) model \( R_{AD}(x, r) = h(x) \), with \( h \) increasing and \( h(0) = 0 \). The shape
of the evaluation function $h$ further controls for loss aversion, diminishing sensitivity etc. KRA and FGL use AD models with $h$ (piecewise) linear. An important criticism to AD models is their failure to capture dynamics of the reference effect in response to changes in the reference price. A natural generalization that satisfies $(DC)$ is $R(x, r) = h(x)s(r)$, with $s$ a positive decreasing scaling function.

Another model used in the literature is the relative difference (RD) model $R_{RD}(x, r) = h(x/r)$ (with $h$ increasing and $h(0) = 0$), motivated by the Weber-Fechner law in psychophysics. This law suggests that consumers perceive price differences in proportional, rather than absolute terms, a theory that is experimentally supported by mental accounting studies, such as the small example in the previous section (that can be explained by consumers not responding to the magnitude of the price differential $r - p = 50$ Euro, but to the percentage difference relative to the base price: $r/p = 50\%$ vs. $4\%$). RD models have been used in conjunction with linear and logarithmic forms for $h$ (Winer 1988), as well as exponentials (Anderson and Rasmussen 2004). A generalization of RD models is $R(x, r) = h(xs(r))$, with $s$ a positive decreasing scaling function. This satisfies $(DC)$ if $h$ is convex increasing, or $h'$ is inelastic to changes in $x$.

While our analysis focuses on the most general reference dependence demand model, we systematically indicate how our general results specialize to the absolute and relative difference models.

3. Dynamic Pricing Model

This section sets up our dynamic pricing model, together with general assumptions on how consumers’ memory works in adjusting reference prices based on price perceptions.
We consider a profit maximizing monopolist who faces a homogeneous stream of repeated customers over an infinite time horizon. The monopolist decides in every period what price $p$ to charge to customers, from a bounded positive interval $P = [0, \bar{p}]$ (our results allow for a non-zero lower bound on price; see the Remark of Section B in the Appendix). The resulting profit is $\Pi(p, r) = (p - c)D(p, r)$, where $r$ is the current reference price and $D(p, r)$ the corresponding demand satisfying Assumption 1. We assume for simplicity and without loss of generality the variable cost to be $c = 0$; all our results hold for a non-zero constant cost $c$.

In each period, customers update their current reference price $r_t$ as they observe the new price $p_t$ charged by the firm. The adaptive expectation framework Nerlove (1958) motivates a reference price that is a weighted average of past prices, where more recent prices are given more weight. Throughout of the paper, we assume that the reference price formation mechanism is given by:

$$r_t = \alpha r_{t-1} + (1 - \alpha) p_{t-1}, \quad \alpha \in [0, 1).$$ (1)

Exponential smoothing is the most commonly used and empirically validated reference price mechanism in the literature (e.g. Winer 1986, Sorger 1988, Greenleaf 1995, KRA, FGL). Some of our results hold for more general mechanisms, including inflation effects and price trends in general (in this case our insights and results remain valid for inflation-adjusted prices). The memory model described by (1) allows for additional insights by quantifying the speed of the memory adjustment process. The memory parameter $\alpha$ captures how strongly the reference price depends on past prices. Lower values of $\alpha$ represent a shorter-term memory; in particular, if $\alpha = 0$ then the reference price equals the past price. Memory is an indicator for shopping frequency, hence can be a proxy for customer loyalty. Actual estimated parameters range from $\alpha = 0$ (Krishnamurthi et al. 1992) to $\alpha = 0.925$ (Greenleaf 1995), much depending on the type of product or service involved.

For an initial reference price $r_0$, we model the firm’s long term profit optimization problem as:

$$V(r_0) = \sup_{p \in P} \sum_{t=0}^{\infty} \beta^t \Pi(p_t, r_t)$$

$$s.t. \quad r_t = \alpha r_{t-1} + (1 - \alpha) p_{t-1}, \quad t \geq 1,$$

where $\beta \in (0, 1)$ is a discount factor. Profit-per-stage is bounded, so the value function is the unique bounded solution of the following Bellman equation (Stokey et al. 1989):

$$V(r) = \sup_{p \in P} W(p, r), \quad \text{where} \quad W(p, r) = \Pi(p, r) + \beta V(\alpha r + (1 - \alpha) p).$$ (2)

A consequence of Assumption 1 is that short term profit $\Pi(p, r)$, and the value function are increasing (Stokey et al. 1989, Theorem 4.7), i.e. larger reference prices are preferable for the firm.
Remark 2. The value function $V(r)$ is an increasing function of the reference price $r$.

Because the action set $P$ is compact and all functions are continuous, there exists an optimal stationary pricing policy that solves Problem (2), and associates with any given reference price $r$, one or more optimal prices. We define the optimal pricing policy as $p^*(r) = \arg \max_p W(p, r)$. To avoid the ambiguity of multiple solutions, $\arg \max_x f(x)$ refers without loss of generality to the largest maximizer of the function $f$. The optimal reference price policy $q^*(r) = \alpha r + (1 - \alpha)p^*(r)$ provides the next reference price associated with the optimal pricing policy.

The pricing path $\{p^*_t\}$ starting with an initial state $r_0$ is computed sequentially as $p^*_t = p^*(r^*_t)$, where the state variable is updated in each period via the reference price formation mechanism $r^*_t = q^*(r^*_t - 1)$. The sequence $\{r^*_t\}$ of optimal reference prices is referred to as the state path; for the consumer, this is the sequence of perceived prices or price expectations. A steady state $p^{**}$ for Problem (2) is one from which it is suboptimal to deviate, i.e. $p^*(p^{**}) = q^*(p^{**}) = p^{**}$, so the optimal price path starting at $r_0 = p^{**}$ is constant.

Our goal is to investigate when such a steady state exists and to characterize stability and monotonicity properties of the price and reference price paths. These results critically depend on consumer’s reaction to discounts and surcharges, captured by loss aversion.

4. Model without Loss Aversion

This section focuses on the case when consumers are loss neutral (LN), i.e. the reference effect is smooth; loss seeking behavior is investigated at the end of this section. We first characterize the unique steady state, and then investigate stability and monotonicity properties of the price and reference price strategies, under very general modeling assumptions. Two additional assumptions are made throughout Sections 4.1-4.3.

Assumption 2 (Smooth Reference Effect). $R(x, r)$ is twice differentiable in $x$ and $r$.

The slope of the reference effect at $x = 0$ is denoted $\lambda(r) = R_x(0, r)$. Our results require some typical technical assumptions on the short term profit $\Pi(p, r) = \pi(p) + \Pi^R(p, r)$. Here $\pi(p) = \Pi(p, p) = pD(p, p)$ is the profit in a market where consumers don’t form reference effects ($R \equiv 0$), referred to as base profit, and $\Pi^R(p, r) = pR(r - p, r)$ is the profit from the reference effect, or reference profit.

Assumption 3 (Profit). (a) $\pi(p)$ is non-monotonic and concave in $p$. (b) $\Pi_p(r, r) = \pi'(r) - r\lambda(r)$ is strictly decreasing in $r$. (c) $\Pi^R(p, r)$ is concave in $p$ and supermodular in $(p, r)$.

Some intuition and motivation for these assumptions is provided next. Non-monotonicity of $\pi(p)$ states that the firm is not constrained to charge sub-optimal prices (in a market without reference effects). This will allow to rule out pathological boundary steady states.
Part (b) states that, if the firm charges the reference price, a small change in price has higher impact on profit at lower reference prices. That is, marginal profit diminishes as price is set to the reference price. Strict monotonicity is necessary to insure uniqueness of the steady state. A sufficient condition for Assumption 3 (b), given (a), is $r\lambda(r)$ strictly increasing in $r$.

Diminishing marginal profit (concavity) is a typical economic assumption. Concavity of $\pi$ and $\Pi^R$ implies concavity of short term profit $\Pi$. The supermodularity (see p.7) assumption (c) states that marginal short term profit increases with the reference price. This is consistent with modeling assumptions used in the literature, and summarized by Winer (1988) (those demand functions are typically additively separable, in particular, linear, in price and reference price). Various sets of sufficient conditions are provided in Section 5.3. One is supermodularity of demand $D(p,r)$, which in particular implies that demand is more elastic at higher reference prices. In support of this assumption, Kamen and Toman (1970) found evidence that with higher reference prices, consumers were more likely to switch brands of gasoline, for a given price differential.

4.1. Myopic Strategies for Loss Neutral Buyers

A firm that ignores the effect of current prices on future demand will focus on maximizing short term profit. Intuitively, such a myopic firm should charge higher prices as consumers’ price expectations are higher. This is insured by profit supermodularity (Assumption 3 (c)), via Topkis’ Theorem (Topkis 1998, Theorem 2.8.2 implies $x^*(y) = \arg \max_{x,y} f(x,y)$ increasing in $y$ whenever $f$ is supermodular).

Lemma 1. The single period profit maximizing price $\tilde{p}(r) = \arg \max_{p \in P} \Pi(p,r)$ is increasing in $r$.

Formally, the problem solved by a myopic firm is equivalent to setting $\beta = 0$ (full discounting) in Problem (2). Consider two firms, one myopic, the other strategic (optimizer), starting with the same initial reference price $r_0$, and applying sequentially their respective pricing policies, while consumers’ memory evolves according to the same $\alpha$-smoothing mechanism. The state of the system at time $t$, for the myopic and optimizing firms are given by $r_t^M$ and $r_t^*$, respectively. The corresponding myopic, respectively optimal prices charged at time $t$ are $p_t^M = \tilde{p}(r_t^M)$ and $p_t^* = p^*(r_t^*)$. We show that the myopic firm systematically underprices, following a monotonic pricing strategy. The results are driven by supermodularity of short term profits (Assumption 3 (c)).

Proposition 1 (Myopic vs. Strategic Firm). For any reference price $r$, the myopic firm underprices the product, i.e. $p^*(r) \geq \tilde{p}(r)$. Furthermore, for any initial reference price $r_0$, the price charged by the myopic firm at any point in time is less than the optimal price, i.e. $p_t^M \leq p_t^*$, for all $t$. Moreover, all myopic price paths monotonically converges to a unique constant price $p^{M*}$. 
The prices charged by the myopic firm are oblivious of their eroding effect on future demand, hence future profits. By charging higher prices, the optimizing firm trades off current profits for future long-term profitability from higher reference prices. Figure 2 illustrates these results; in this case, the observed difference between optimal and myopic profits is about 10%. Throughout the paper, numerical examples are calculated using the Compecon Matlab toolbox developed by Miranda and Fackler (2002).

Figure 2    Myopic vs. optimal pricing policies; \( D(p,r) = 1 - 2p + \frac{r}{p+\varepsilon} \), \( P = [0.1, 0.3] \), \( \alpha = 0.80 \), \( \beta = 0.95 \).

Figure 2 also suggests that the optimal pricing paths may converge monotonically to a single steady state price. These issues are investigated next.

4.2. Steady State for Loss Neutral Buyers

Define the profit optimizing price in absence of reference effects \( \tilde{p} = \arg \max_{p} \pi(p) \). The Bellman equation (2) implies that, if a steady state \( p^{**} \) exists, then \( V(p^{**}) = \frac{\pi(p^{**})}{1-\beta} \), whereas generally \( V(r) \geq \frac{\pi(r)}{1-\beta} \) for all \( r \in P \). Assumption 3 (a,b) insures that such a steady state is unique and can be computed as the solution of a simple implicit equation:

**Theorem 1 (Steady State).** If Problem (2) admits a steady state \( p^{**} \), then this is the unique solution of:

\[
\frac{\pi'(p)}{1-\beta} = \frac{p\lambda(p)}{1-\alpha\beta}.
\] (3)

Furthermore, \( p^{**} \) is continuously decreasing in \( \alpha \), increasing in \( \beta \), and satisfies \( p^{M*} \leq p^{**} \leq \tilde{p} \).
At steady state, a small increase in price $\Delta p$ leads to a marginal increase in the long-term base profit equal to $\pi'(p) \Delta p$. However, on the transition, customers perceive a loss and reduce their demand. The resulting profit loss is $p\lambda(p)\Delta p$ in the first period, $\alpha p\lambda(p)\Delta p$ in the second period and so on. Hence the implicit equation (3) reflects the balance between maximizing long-term profits (without reference effects) and the discounted transitory costs due to the reference effect.

Equation (3) confirms that myopic firms ($\beta = 0$) set lower steady state prices $p^{M*}$. On the long run, strategic firms should charge lower prices when consumers form reference effects, than when they do not, i.e. $p^{**} \leq \tilde{p}$, approaching it as $\beta \to 1$.

In particular, Theorem 1 obtains the following closed form expressions for the corresponding steady states for absolute and relative difference models discussed in Section 2.3. This generalizes the FGL result for linear reference effects.

**Corollary 1 (Steady State for AD and RD Models).** Consider the problem with linear base demand $D(p, p) = b - ap$, and reference effect $R_{AD}(x, r) = h(x)$, respectively $R_{RD}(x, r) = h(x/r)$. The corresponding steady states, if they exist, are $p^{**}_{AD} = \frac{b}{2a + \Lambda}$ and respectively $p^{**}_{RD} = \frac{b - \Lambda}{2a}$, where $\Lambda = h'(0)\frac{1-\beta}{1-\alpha\beta}$.

**Consumer Heterogeneity and Sensitivity Analysis Insights.** Consider a heterogeneous market, where consumer types differ in shopping frequency, captured by the memory parameter $\alpha_i$, and sensitivity to past prices, i.e. the reference effect $R_i$. Demands without reference effects are assumed identical across segments (i.e. $D_i(p, p) = D(p, p)$).

If the firm can segment the market, Theorem 1 indicates that on the long run, each segment should be charged a steady state price $p^{**}_i$ that solves $rac{\pi'(p)}{p(1-\beta)} = \frac{\lambda_i(p)}{1-\alpha_i\beta}$. The steady state price only depends on the reference effect via its slope at zero, specifically, it decreases with the magnitude of $\lambda(\cdot)$. That is, if consumers share the same memory parameter $\alpha$, those who are more sensitive to past prices will be charged lower prices on the long run.

**Remark 3.** If $\alpha_1 = \alpha_2$ and $R_2$ is a “stronger” reference effect than $R_1$, i.e. if $\lambda_1(r) \leq \lambda_2(r)$ for all $r$, then the corresponding steady state prices, if they exist, satisfy $p^{**}_1 \geq p^{**}_2$.

On the other hand, if consumers only differ in their memory parameter $\alpha_i$, then segments with higher $\alpha_i$ are charged lower steady state prices, all else equal. While charging lower prices to loyal customers (proxied by higher $\alpha_i$) is in line with current practices in consumer goods industries, our results show that it may not be the optimal strategy if these shoppers are less sensitive to price changes, as measured by $\lambda_i$. In general, the firm should charge lower prices to segments with higher
long run marginal sensitivity to the reference effect, as measured by \( \lambda_i(z)/(1-\alpha_i\beta) \). Specifically, the same argument as Remark 3 shows that \( p_i^* \geq p_j^* \), if \( \frac{\lambda_i(r)}{1-\alpha_i\beta} \leq \frac{\lambda_j(r)}{1-\alpha_j\beta} \) holds for all \( r \).

Finally, for comparison purposes, consider the case where the firm cannot segment the market and offers the same price to all segments. We show in Section D of the Appendix that the common steady state price (if one exists) solves

\[
\pi'(p) = p \sum_{i=1}^N \frac{\lambda_i(p)}{1-\alpha_i\beta}, \text{ and is bounded between the lowest and highest prices charged to each consumer type in a perfectly segmented market } p^* \in [\min p_i^*, \max p_i^*].
\]

### 4.3. Stability and Monotonicity with Loss Neutral Buyers

This section investigates the existence and stability of the unique steady state price characterized in Theorem 1. We show global convergence of the optimal prices to steady state, and study local and global monotonicity properties of the state and optimal price paths.

We start with the most general results that require no further assumptions. Recall that the reference price policy was defined by

\[
q^*(r) = \alpha r + (1-\alpha)p^*(r).
\]

**Lemma 2 (Monotone Reference Price Policy).** The reference price policy \( q^*(r) \) increases in \( r \).

The result follows from Assumption 3(c), and implies the following:

**Theorem 2 (Global Stability and State Path Monotonicity).** Problem (2) admits a unique steady state \( p^* \), characterized by Theorem 1, to which all optimal price paths converge. All corresponding reference price paths converge to \( p^* \) monotonically.

Under the conditions of Theorem 2, the monopolist’s strategy involves decreasing customer’s price expectations by always pricing below the current reference price, or vice versa. If the reference price is high, the optimal pricing policy is a skimming-like strategy: the firm starts with a relatively high introduction price, then always prices below the reference price \( (r_t > p_t) \). In this case, consumers perceive in each period a gain relative to their expectations, which stimulates demand, and hence profits in the current period. Low reference prices lead to a penetration-type strategy, whereby the firm always prices above the customers’ reference price \( (r_t < p_t) \). The loss in current profit from a negative reference effect, is outweighed (1) by a higher short-term profit without reference effect, \( \pi(p_t) \), and (2) by the increase in future profits due to higher reference prices in the next period.

Theorem 2 does not guarantee monotonicity of the corresponding pricing paths. This holds whenever the pricing policy is monotonic. Some sufficient conditions are provided next. Recall that a real-valued \( C^2 \) function \( f(x, y) \) is strongly concave if there exist positive constants \( a \) and \( b \) such that \( f(x, y) + ax^2 + by^2 \) is concave.
Lemma 3 (Monotone Price Policy). Each of the following alternative conditions is sufficient for the optimal pricing policy \( p^*(r) \) in Problem (2) to be increasing in \( r \):

(a) Consumers have short memory \( \alpha = 0 \), i.e. \( r_{t+1} = p_t \) for all \( t \);

(b) Demand \( D(p,r) \), or equivalently, the reference effect \( R(r - p, r) \), is convex in \( r \);

(c) Short term profit \( \Pi(p,r) \) is strongly concave on the interior of its domain, and there exists a global constant \( K > 0 \) such that \( \Pi_{rr}(p,r) \geq -K \) and \( \Pi_{pr}(p,r) \geq K/2 \).

The result follows from supermodularity of \( W(p,r) \), the function on the right hand side of the Bellman equation (2). This reduces to supermodularity of short term profit (Assumption 3 (c)) under condition (a). Condition (b) achieves this by additionally insuring convexity of short term profit in \( r \), and hence of the value function. Condition (c) insures a uniform lower bound on the degree of concavity of the value function \( V \), that is compensated by supermodularity of short term profits. At the limit, for \( K = 0 \) we recover the conditions of part (b). Strong concavity is a technical condition insuring second order differentiability of the value function.

Theorem 3 (Global Price Monotonicity). Under the conditions of Lemma 3, all optimal pricing paths for Problem (2) converge monotonically to the unique steady state price \( p^{**} \), characterized by Theorem 1.

Figure 3 Optimal pricing paths for \( D(p,r) = e^{-p} + 0.5(e^{0.5(r-p)} - 1), \alpha = 0.8, \beta = 0.9 \).

In particular, we obtain the price monotonicity results of KRA and FGL for linear AD models \( R_{AD}(x,r) = \lambda x \); these satisfy part (b) of Lemma 3. Part (c) of Lemma 3 (but not (b)) obtains price
monotonicity for linear RD models $R_{RD}(x,r) = \lambda x/r$, on a price range $P = [\bar{p}, \bar{p}]$ with $\bar{p}/\bar{p} \leq \sqrt{2}$, i.e. price dispersion not exceeding 18%.

Theorem 3 insures that only three possible pricing strategies are optimal: skimming (if initial price expectations are high, start with a high price and then keep decreasing it), penetration (if initial price expectations are low, start with a low price, then increase it), or constant pricing. Moreover, after a transient regime, the optimal prices stabilize to a steady state price. The results follow from the monotonicity of the policy function (Lemma 3). This also insures that the price paths do not cross, i.e. any increase in the initial reference price shifts the entire pricing sequence upwards. These results are illustrated with an example in Figure 3.

Furthermore, in a neighborhood of the steady state, prices are eventually monotonic, i.e. local price monotonicity holds. Strong concavity of short term profits is a technical condition that insures differentiability of the value function and pricing policy.

**Proposition 2 (Local Price Monotonicity).** If $\Pi(p,r)$ is strongly concave on the interior of its domain, then all optimal pricing paths are locally monotonic around the steady state.

Local price monotonicity implies that, on the long run, the slope of the reference price policy $q^*(r) = \alpha r + (1 - \alpha)p^*(r)$ is steeper than $\alpha$. That is, reference prices converge to steady state at a rate that is ultimately faster than the memory parameter $\alpha$.

**Transient High-Low Pricing.** At the introductory stage, however, the monopolist may have an incentive to increase reference prices at a slower rate, achieved by high-low pricing. Such a strategy, depicted in Figure 4, contrasts the insights obtained by KRA and FGL under linear demand models. The optimal pricing policy $p^*(r)$ and price path $\{p^*_t\}$ are non-monotonic under the conditions of Theorem 2 and Proposition 2, but violating Lemma 3. The pricing strategy illustrated in Figure 4 is consistent for example with catastrophe insurance pricing in the U.S. before major hurricanes, when consumers reference prices are much lower than profit optimizing prices (we thank Paul Kleindorfer for suggesting this example; see endnote 1. for intuition).

This example illustrates that high-low pricing is technically possible in the loss neutral case, contrary to previous insights in the literature; similar behavior can be observed for RD models. Nevertheless, extensive numerical simulations revealed that the optimal pricing path is typically monotonic; at most, it displays a relatively small direction swing from decreasing to increasing prices, as in Figure 4 (simulated models for $R(x,r)$ include $xe^{-ar}$, $x/r^k$, $1 - e^{-ax}$, $e^{ax/r^k} - 1$, $a,k > 0$ for a variety of parameters satisfying Assumptions 1-3). So it is fair to say that, if consumers are loss neutral, we do not find strong numerical support for transient high-low pricing, but cannot rule it out either.
4.4. Loss Seeking Buyers

High-low pricing is provably optimal if consumers are focused on gains, i.e. loss seeking (LS). In this case the optimal pricing policy for Problem (2) cycles. Consumers are loss seeking if they respond more to discounts than to surcharges, i.e. if the reference effect has a convex kink at zero. This type of asymmetry is inconsistent with prospect theory, but has found some empirical validation in the marketing literature (e.g. Krishnamurthi et al. 1992, Greenleaf 1995). We address it for completeness; only Assumption 1 stands in place for the next result:

**Proposition 3 (Cyclic Policy).** If buyers are loss seeking, i.e. the reference effect $R(x, r)$ satisfies (LS), then Problem (2) admits no steady state.

Proposition 1 in KRA obtains a similar result when $R(x, r) = h(x)$ is piecewise linear in $x$.

Proposition 3 suggests that in order for a steady state price to exist, the reference effect $R$ must be smooth, or kinked in the direction predicted by prospect theory. This case is analyzed next.

5. Model with Loss Aversion

This section investigates the transient and long term behavior of the optimal dynamic pricing policy when reference effects are kinked in the spirit of prospect theory. We first set up the model; two additional assumptions on the reference effect, respectively profit are made throughout this section (replacing Assumptions 2 and 3 for the loss neutral case).

5.1. Model and Assumptions

At the aggregate level, prospect theory predicts that the reference effect is captured by an S-shaped function of the reference gap $x = r - p$, with a concave (loss averse) kink at $x = 0$ (Kahneman and
The following assumption, made throughout this section, formalizes the technical aspects relevant for our purposes. An illustration is provided in Figure 5. Recall that a function $f(x)$ is single-crossing if there exists a unique $x_0$ in its domain such that $f(x_0) = 0$. We denote $1_X$ the indicator function of the set $X$.

**Figure 5** An example of kinked reference effect $R_K$.

Assumption 4 (Kinked Reference Effect). The kinked reference effect is given by

$$R_K(x, r) = 1_{(x \geq 0)}R_G(x, r) + 1_{(x \leq 0)}R_L(x, r), \quad (4)$$

where $R_L(x, r)$ and $R_G(x, r)$ are twice differentiable, increasing in $x$ with $R_G(x, r) - R_L(x, r)$ single-crossing in $x$, and satisfying $\lambda_L(r) > \lambda_G(r)$ for all $r$.

Assumption 4 is a weaker formalization of prospect theory ($PT$), as detailed in Section 2.1, including reference dependence ($RD$), and the weaker form of loss aversion ($LA$) in terms of the index $\rho(r) = \frac{\lambda_L(r)}{\lambda_G(r)}$, proposed by Köbberling and Wakker (2005). We do not specifically assume diminishing sensitivity ($DS$) (i.e. concavity/convexity of the gain/loss components), as it is immaterial for our analysis. The single crossing condition allows to write the kinked reference effect as the minimum of the loss and gain counterparts on the relevant domain:

$$R_K(x, r) = \min(R_G(x, r), R_L(x, r)). \quad (5)$$
This condition is unrestrictive in the context of Assumption 4 alone, because the loss and gain parts of $R_K$ can always be smoothly extended so as to satisfy Eq. (5).

Based on (5), the profit per-period can be written as

$$\Pi_K(p, r) = \min(\Pi_G(p, r), \Pi_L(p, r)),$$

(6)

where $\Pi_G(p, r) = \pi(p) + pR_G(r - p, r)$ and $\Pi_L(p, r) = \pi(p) + pR_L(r - p, r)$. In line with our previous assumptions on profit, the following technical assumption is made in this section:

**Assumption 5 (Gain/Loss Profit).** The short term profits $\Pi_G(p, r)$ and $\Pi_L(p, r)$ associated with the smooth reference effects $R_G$, respectively $R_L$, satisfy Assumption 3.

Section 5.3 provides sufficient modeling conditions on the reference effect for Assumptions 4 and 5 to hold, in particular when $R_K$ is convex-concave, as predicted by prospect theory. In particular, it provides concrete ways to construct $R_L, R_G$ that satisfy Assumption 3, and so that Eq. (5) holds.

The corresponding Bellman equation for the problem with kinked demand effect is:

$$(K) \quad V_K(r) = \sup_p \{\Pi_K(r, p) + \beta V_K(\alpha r + (1 - \alpha)p)\}.$$ (7)

5.2. Steady State, Stability and Monotonicity

We extend the results obtained in Section 4 for the loss neutral case, to infer existence and stability properties of the steady state for loss averse buyers. A key role in our analysis here is played by the problem with smooth reference effect:

$$R_\theta(x, r) = \theta R_G(x, r) + (1 - \theta) R_L(x, r).$$ (8)

Denoting the corresponding short term profit $\Pi_\theta(p, r) = \pi(p) + pR_\theta(r - p, r)$, this problem is

$$(P_\theta) \quad V_\theta(r) = \sup_p \{\Pi_\theta(p, r) + \beta V_\theta(\alpha r + (1 - \alpha)p)\}.$$ (9)

For any $\theta \in [0, 1]$, Problem $(P_\theta)$ satisfies our assumptions for the loss neutral case, so all the results of Section 4 apply. Moreover, Eq. (5) implies

$$R_K(x, r) = \min(R_G(x, r), R_L(x, r)) \leq R_\theta(x, r), \quad \theta \in [0, 1].$$ (10)

The same relationship between short term profits, and the corresponding value functions is implied. The proof details of the following result are in the Appendix:

**Lemma 4.** For any $\theta \in [0, 1]$, Problem $(P_\theta)$ satisfies Assumptions 1-3. Its value function satisfies $V_\theta(r) \geq V_K(r)$ for all $r$, and its steady state $p_\theta^{**}$ is also a steady state for Problem $(K)$. 

By loss aversion and construction (8), the slopes of the reference effects $R_L, R_\theta$ and $R_G$ satisfy $\lambda_L(r) \geq \lambda_\theta(r) \geq \lambda_G(r)$. Remark 3 further implies that the steady state prices of the corresponding problems satisfy $p_L^{**} \leq p_\theta^{**} \leq p_G^{**}$. Moreover, as $\theta \in [0, 1]$, the steady states $p_\theta^{**}$ span the interval $[p_L^{**}, p_G^{**}]$.

We are now ready to characterize the set of steady states for general non-linear loss-averse demand effects, and their stability properties, as follows:

**Theorem 4 (Steady State, Stability and Monotonicity under Loss Aversion).** The set of steady states for Problem $(K)$ equals $[p_L^{**}, p_G^{**}]$, so for an initial reference price $r_0 \in [p_L^{**}, p_G^{**}]$, the optimal price path is constant $p_t^* = r_0$. For an initial reference price $r_0 < p_L^{**}$, respectively $r_0 > p_G^{**}$, the reference price path $\{r_t^*\}$ converges monotonically to $p_L^{**}$, respectively $p_G^{**}$, and the corresponding optimal price path $\{p_t^*\}$ also converges to the same steady state.

Our analysis, based on bounding the kinked reference effect, and hence corresponding value function, by the full range of convex combinations of the smooth loss and gain counterparts, suggests a general approach for obtaining structural properties and global stability in dynamic programs with kinked reward structure. A similar approach is used by FGL in their analysis of kinked linear demand models; their results are driven by the explicit solution of the linear demand model, while ours rely on structural convexity arguments.

An illustration of the steady states, pricing strategies and corresponding value function under loss aversion is provided in Figure 6. While technically, high-low pricing is possible within the assumptions of Theorem 4 (e.g. using the model of Figure 4 for $R_L$), numerical results suggest that with most common demand models, prices are typically monotonic. The price monotonicity results obtained in the loss neutral case extend as follows:

**Proposition 4 (Price Monotonicity under Loss Aversion).** If the problems with reference effects $R_G$ and $R_L$ also satisfy the conditions of Lemma 3, respectively Proposition 2, then the price paths for Problem $(K)$ are globally, respectively locally monotonic. In particular, if consumers have short term memory ($\alpha = 0$), then prices are globally monotonic.

Furthermore, the strategy of a myopic firm under loss aversion resembles that for the loss neutral case, except that there is a range of myopic steady states. The proof of the next result follows the combined spirit of Lemma 4, Theorem 4 and Proposition 1.

**Proposition 5 (Myopic Prices under Loss Aversion).** Given any initial reference price $r_0$, the myopic price path $\{p_t^M\}$ lies below the optimal price path $\{p_t^*\}$ and converges monotonically to a myopic steady state. Furthermore, the set of myopic steady states equals $[p_L^{M*}, p_G^{M*}]$, where $p_L^{M*}$ and $p_G^{M*}$ are the steady states for a myopic firm under reference effect $R_L$, respectively $R_G$. 
While the myopic firm systematically underprices, under loss aversion, its policy may be optimal for a range of initial reference prices. For example, consider a piecewise linear AD model with \( D(p, p) = 1 - p, \lambda_L = 0.5, \lambda_G = 0.2, \bar{p} = 0.6 \) and \( \alpha = 0.95, \beta = 0.90 \). We obtain \([p^{M^*_L}, p^{M^*_G}] = [0.400, 0.455]\) and \([p^{M^*_L}, p^{M^*_G}] = [0.426, 0.468]\). In this case, if \( r_0 \in [0.426, 0.455] \), the myopic (constant) pricing strategy and the optimal one coincide.

5.3. Special Demand Models.

The assumptions underlying Theorem 4 impose additional, implicit conditions on the demand model. These involve smoothly extending the loss and gain component of a given kinked reference effect over the entire domain so as to satisfy Eq. (5), and verifying the technical profit Assumption 3 on each component. This section provides explicit modeling conditions on the reference effect that are sufficient to validate these assumptions. We focus on reference effects satisfying the prospect theory claims motivated in Section 2, including diminishing sensitivity (DS) and decreasing curvature (DC). We further infer specific implications for AD and RD models. Finally, we explain how our results extend the ones in the literature, obtained for linear AD models.

5.3.1. Prospect Theory Models. Given a kinked S-shaped reference effect \( R_K(x, r) \), as predicted by prospect theory, the challenge in satisfying Assumption 4 is to smoothly “extend” the convex (loss) and concave (gain) parts over the entire domain so as to satisfy Eq. (5). One such construction involves a linear extension: \( R_L(x, r) = x\lambda_-(r), x \geq 0, R_G(x, r) = x\lambda_+(r), x \leq 0 \), where \( \lambda_-(r), \lambda_+(r) \) denote the left and right \( x \)-derivatives of \( R_K(x, r) \) at \( x = 0 \). This construction satisfies Eq. (5) on \( P \) as long as \( (r - \bar{p})\lambda_+(r) \geq R_L(r - \bar{p}, r) \), where \( \bar{p} \) is the upper bound on price. An alternative extension of the concave part that always satisfies Eq. (5) is \( R_G(x, r) = R_K(x, r)/\rho(r), x \leq 0, \rho(r) \)
where $\rho(r) = \frac{\lambda_r(r)}{\kappa_r(r)} > 1$. The smooth extensions $R_L, R_G$ are further conditioned to satisfy Assumption 3 (intuitively motivated in Section 4). Sufficient conditions for profit concavity and supermodularity are provided in the next lemma.

**Lemma 5.** Assumption 3 (c) holds if the reference effect $R$ satisfies (LN) and (RD), and either:

1. $R(r - p, r)$ is supermodular in $(p, r)$ and $p \frac{R_{xx}(r - p, r)}{R_x(r - p, r)} \leq 2$; or
2. $R(x, r)$ is concave in $x$ and satisfies (DC); or
3. $R(x, r)$ satisfies (DC) and the following two additional conditions:
   - (a) The marginal effect of discounts/surcharges $R_x$ is price inelastic, i.e. $p R_{xx}(r - p, r) R_x(r - p, r) \leq 1$; and
   - (b) The marginal reference effect shift $R_r$ is price elastic, $p R_{xr}(r - p, r) R_r(r - p, r) \geq 1$, when $R_r(r - p, r) < 0$.

Section 4 provides arguments and motivation for supermodularity of the reference effect in $(p, r)$. The other technical conditions are bounds on the price-elasticity of the marginal reference effect.

We obtain the following self-contained result for kinked S-shaped reference effects:

**Proposition 6 (Convex-Concave Reference Effects).** Theorem 4 holds if $\pi(p)$ is concave and non-monotonic, and the reference effect $R_K(x, r)$ satisfies (RD), (DS), (DC) and:

- (a) $R_K(x, r)$, $x \leq 0$, satisfies the conditions of Lemma 5 (in particular it is linear in $x$);
- (b) $R_K(r - \bar{p}, r) \leq (r - \bar{p}) \lambda_+(r)$, for all $r \in P$, or $(b') \rho(r) = \frac{\lambda_r(r)}{\kappa_r(r)} > 1$ is increasing in $r$;
- (c) $r \lambda_+(r)$ and $r \lambda_-(r)$ are strictly increasing in $r$.

The additional specifications of Proposition 6 are also supported by behavioral evidence. A significant loss aversion index ($\rho \approx 2$, Ho and Zhang 2005) justifies (b) on a limited price domain; (b) implies (LA). Alternatively, (b’) states that the index of loss aversion is increasing in $r$, in particular constant. Empirical evidence suggests risk neutrality for (small) losses, implying that the convex part is actually linear in $x$, $R_K(x, r) = x \lambda_-(r), x \leq 0$. In this case, condition (a) holds and condition (b) is precisely (LA): $\lambda_+(r) \leq \lambda_-(r)$. Empirical support for (DC) is discussed in Section 2. Kahneman and Tversky (1979, p.277) suggest a slow variation of the reference effect with the reference level $r$, which supports (c) (i.e. $\lambda(r)$ decreases slower than $1/r$).

### 5.3.2. Absolute and Relative Difference Models.

The conditions of Theorem 4 hold for AD and RD reference effects $R_{AD}(x, r) = h(x)$ and $R_{RD}(x, r) = h(x/r)$, discussed in Section 2.3, as long as the corresponding assumptions on profit are satisfied. For both models, the only condition that effectively needs to be verified is profit supermodularity, which implies concavity of reference profits in price (joint concavity for RD models). For both models, supermodularity always holds for the concave part (by part 2. of Lemma 5), so additional technical conditions concern only the
convex part. In particular, the results of Theorem 4 hold if the reference effect is linear in the loss domain, e.g. for concave-kinked linear AD and RD models. The following result is a direct consequence of Proposition 6, with Proposition 4 implying price monotonicity in the AD case.

**Corollary 2 (Kinked Linear AD and RD Models).** Assume that \( \pi(p) \) is concave and non-monotonic. The results of Theorem 4 and Proposition 2 hold for linear AD and RD-models with a loss-averse kink; the corresponding steady states are characterized in Corollary 1. In addition, global price monotonicity obtains for the AD case, or for \( \alpha = 0 \).

### 5.3.3. Linear models. Comparison with the literature.

FGL and KRA provide special cases of our results for AD models with linear base demand \( D(p, p) = b - ap \) and (kinked) linear reference effects \( R_G(x, r) = \lambda_G x, R_L(x, r) = \lambda_L x, R_K = \min(R_L, R_G) \). FGL characterize the steady state and prove monotonicity of the pricing path in the loss neutral case (\( \lambda_G = \lambda_L \)), by solving the Euler equation explicitly in continuous time (their Proposition 1). They further use these results in a two stage approach to characterize (in their Proposition 2) the pricing policies under loss aversion (\( \lambda_G < \lambda_L \)). Price monotonicity was conjectured in KRA, who proved it in the loss neutral case and for \( \alpha = 0 \) (their Proposition 2). KRA also showed that in the loss seeking case (\( \lambda_G > \lambda_L \)) the optimal policy cycles (their Proposition 1).

We generalize these results under general reference dependent demand models as follows. A closed form characterization for the steady state is given in Theorem 1 and Corollary 1. Global stability is proved to generally hold in the loss neutral case (Theorem 2), as well as in the loss averse case (Theorem 4, Proposition 6 and Corollary 2). While price monotonicity holds on the long run (Proposition 2), it does not extend to the general case; general sufficient conditions extending FGL and KRA are identified in Theorem 3 (loss neutral), Proposition 4 and Corollary 2 (loss averse). Finally, our Proposition 3 shows that the policy cycles under very general loss seeking reference dependent demand models.

### 6. Conclusions

In this paper we considered the dynamic pricing problem of a monopolist in a market for a frequently purchased product or service, where demand is a function of current and past prices, summarized by an internal reference price. The reference price is continuously adjusted and acts as an anchoring standard against which current prices are compared, under a behaviorally validated mechanism. We extended the state of the literature on dynamic pricing with reference price effects by (1) providing very general non-linear reference dependent demand models, that capture dynamics in the reference effect as the reference price shifts (which previous, mainly linear models
did not), and (2) obtaining structural (as opposed to parametric) results for these general demand models.

Our results indicated that managers who ignore long term implications of their pricing strategy (due to consumer memory and anchoring effects), will consistently price too low, thereby systematically losing revenue.

We also showed that high-low pricing can be explained by consumers reacting more to discounts than surcharges (loss seeking). Otherwise, the firm’s optimal pricing strategies converge over the long run to a constant price, characterized by a simple, insightful implicit equation. The value of the steady state price decreases with customers’ memory and with their sensitivity to the reference effect, all else equal. Higher loss aversion leads to more pricing flexibility, i.e. a wider range of steady state prices.

Dependence of demand on past prices determines prices to vary over time, and in particular leads to skimming and penetration type strategies. We showed that in general customers’ perception of prices follows a monotonic trajectory. If customers have short term memory or they are increasingly sensitive to reference prices, then the pricing trajectory is also monotonic. While high-low pricing is optimal in certain cases, our extensive numeric simulations found monotonic prices to be most typical.

Several immediate extensions are foreseeable, but beyond the scope of this paper. (1) While our work relies on a deterministic model, it would be interesting to investigate the robustness of our findings with respect to the firm’s uncertainty about customer’s memory and sensitivity to past prices. High-low pricing, as observed in the market place, is likely to be a result of the firm’s exploratory behavior under such uncertainties. (2) The results of FGL indicate that (Cournot) competition does not alter the qualitative behavior of the optimal strategy for linear demand models. Whether this remains true under our general assumptions, and for different models of competition is an open question. (3) The “Lucas Critique” argues that expectation parameters and endogenous variable dynamics depend on policy parameters. In our context, this means that as soon as a firm tries to exploit this model, the empirically estimated consumer behavior model is likely to change. It would be interesting to conduct an empirical investigation to model this feedback process, and further incorporate it in the analysis. (4) Finally, it is natural to expect that consumers’ strategic behavior will interact with the optimal strategies resulting from our analysis. For example, when prices decline or the optimal policy cycles, the firm will be vulnerable to consumers learning to wait for lower future prices. Strategic firms should take these effects into
account when establishing their pricing policies. We intend to incorporate some of these effects in future work.

Appendix. Proofs.

A. Proof of Proposition 1

By Remark 2, $V$ is increasing, implying that $p^*(r) = \arg\max_{p \in P} \{\Pi(p, r) + \beta V(\alpha r + (1 - \alpha)p)\} \geq \arg\max_{p \in P} \Pi(p, r) = \tilde{p}(r)$.

The rest of the proof is by induction. Given the same initial state $r_0$ we have $p_0^* = p^*(r_0) \geq \tilde{p}(r_0) = p_0^M$. Suppose that $p_{i-1}^* \geq p_{i-1}^M$ and $r_{i-1}^* \geq r_{i-1}^M$. The memory equation (1) implies $r_i^* \geq r_i^M$. Moreover $\tilde{p}(r)$ is increasing in $r$ by Lemma 1, so $p_i^M = \tilde{p}(r_i^M) \leq \tilde{p}(r_i^*) \leq p^*(r_i^*) = p_i^*$. Therefore the myopic policy always underprices.

Suppose that $p_0^M = \tilde{p}(r_0) \geq r_0$, so $r_1^M \geq r_0$ (the other case is symmetric). Lemma 1 implies $p_1^M \geq p_0^M$. Forward induction yields an increasing sequence $\{p_t^M\}$, in the bounded domain $P$. Hence there exists $p^{M*}(r_0) = \lim_{t \to \infty} p_t^M$. A myopic steady state price $p^{M*}$ must solve the first order optimality condition $\Pi_p(p, r) = 0$. The function $\Pi_p(p, r)$ is strictly decreasing in $p$ by Assumption 3 (b), insuring uniqueness of $p^{M*}$.

B. Proof of Theorem 1

The proof follows a variational approach. Suppose there exists an interior steady state $r$ such that $p^*(r) = r$. For small enough $\delta > 0$, consider the feasible pricing policy $\phi(\delta) := \{p_0 = r - \delta; p_1 = r + \alpha\delta; p_t = r, t \geq 2\}$. So the corresponding state path is $r_0 = r, r_1 = r - (1 - \alpha)\delta, r_t = r, t \geq 2$. The value function under policy $\phi(\delta)$ is:

$$V_\delta(r) = \pi(r - \delta) + (r - \delta)R(\delta, r) + \beta\pi(r + \alpha\delta) + \beta(r + \alpha\delta)R(-\delta, r - (1 - \alpha)\delta) + \beta^2 V(r)$$

Here $V(r)$ is the optimal value achieved by the constant pricing strategy $p^*(r) = r$, so we can write $V(r) = \pi(r) + \beta\pi(r) + \beta^2 V(r) \geq V_\delta(r)$. Subtract this from the above to obtain:

$$\pi(r - \delta) - \pi(r) + \beta(\pi(r + \alpha\delta) - \pi(r)) + (r - \delta)R(\delta, r) + \beta(r + \alpha\delta)R(-\delta, r - (1 - \alpha)\delta) \leq 0$$

Dividing by $\delta$, and letting $\delta$ go to zero, we rearrange terms to obtain

$$(1 - \alpha\beta)\pi'(r) \geq r(1 - \beta)\lambda(r).$$

(11)

The right-hand side is derived from $\lim_{\delta \to 0} \frac{R(\delta, r)}{\delta} = \lambda(r)$ and $\lim_{\delta \to 0} \frac{R(-\delta, r - (1 - \alpha)\delta)}{\delta} = -\lambda(r) - (1 - \alpha)R_r(0, r) = -\lambda(r)$, because $R_r(0, r) = 0$ from $R(0, r) = 0$ (Remark 1).
The same argument for the feasible policy \( \varphi(-\delta) \) yields the opposite inequality

\[
(1 - \alpha \beta) \pi'(r) \leq r(1 - \beta) \lambda(r) .
\]

(12)

Combining (11) and (12), an interior steady state must solve:

\[
(1 - \alpha \beta) \pi'(p) = (1 - \beta) p \lambda(p) .
\]

(13)

Note, under additional differentiability assumptions, this result can also be obtained from the Euler equation. The function

\[
G(p, \alpha, \beta) = \frac{1 - \alpha \beta}{1 - \beta} \pi'(p) - p \lambda(p) = \Pi_p(p, p) + \frac{1 - \alpha}{\beta^1 - 1} \pi'(p)
\]

is strictly decreasing in \( p \) because \( \pi \) is concave and \( \Pi_p(p, p) \) is strictly decreasing (Assumption 3.b). This implies uniqueness of the interior steady state price \( p^* \) solving \( G(p^*, \alpha, \beta) = 0 \). Because \( G \) is decreasing in \( \alpha \) and increasing in \( \beta \), the same is true of \( p^* \).

Concavity of \( \pi \) implies that \( V(\tilde{p}) \geq \frac{\pi(\tilde{p})}{1 - \beta} \geq \frac{\pi(p^*)}{1 - \beta} = V(p^*) \). Because \( V \) is monotonic and \( \tilde{p} \) is the largest maximizer of \( \pi \), it follows that \( p^* \leq \tilde{p} \). From Proposition 1, \( p_t^M \leq p_t^* \) implies \( p_t^{M*} = \lim_{t \to \infty} p_t^M \leq \liminf_{t \to \infty} p_t^* = p^{**} \). Therefore, \( p^{**} \) satisfies \( p_t^{M*} \leq p^{**} \leq \tilde{p} \).

It remains to exclude boundary steady states. Non-monotonicity of \( \pi(p) \) implies \( \tilde{p} < \bar{p} \), so \( \bar{p} \) cannot be a steady state. On the other hand, \( p = 0 \) cannot be a steady state because any non-zero pricing strategy achieves positive profits, so \( V(0) > 0 \).

Remark: Our results extend when there is a positive lower bound on price \( p > 0 \), provided that at the lowest reference price \( r = p \), demand \( D(p, p) \) is price elastic at \( p = p \) (i.e. elasticity exceeds one). This is equivalent to \( \Pi_p(p, p) > 0 \) and insures \( p^*(p) > p \), so \( p \) cannot be a steady state.

C. Proof of Remark 3

Let \( G_1, G_2 \) denote the functions in Eq. (14) corresponding to \( \lambda_1, \lambda_2 \) respectively; we omit the arguments \( \alpha, \beta \) for simplicity. Theorem 1 implies \( G_i(p_i^{**}) = 0, i = 1, 2 \) and because \( \lambda_1(r) \leq \lambda_2(r) \), we have \( G_2(p_2^{**}) \leq 0 = G_2(p_2^{**}) \). Because \( G_2 \) is decreasing in \( p \), we obtain \( p_1^{**} \geq p_2^{**} \).

D. Unsegmented Market of Heterogeneous Consumers

Let \( r = (r_1, \ldots, r_N) \) denote the vector of reference prices for the \( N \) consumer types. Our previous assumptions are assumed to hold for each consumer type, with demand modeled as \( D_i(p, r_i) = D_i(p, p) + R_i(r_i - p, r_i) \). The total profit without reference effect is \( \pi(p) = \sum_{i=1}^N \pi_i(p) \), where \( \pi_i(p) = \ldots \)
where $P = \{p, \alpha r + (1 - \alpha) p\}$ and $F(r, q) = \Pi \left( \frac{q - \alpha r}{1 - \alpha} \right)$, we have:

\[
F(r', q') - F(r', q) = \Pi \left( \frac{q' - \alpha r'}{1 - \alpha} \right) - \Pi \left( \frac{q - \alpha r'}{1 - \alpha} \right)
\]
\[ \geq \Pi\left(\frac{q - \alpha r'}{1 - \alpha}, r\right) - \Pi\left(\frac{q - \alpha r'}{1 - \alpha}, r\right) \]
\[ \geq \Pi\left(\frac{q - \alpha r}{1 - \alpha}, r\right) - \Pi\left(\frac{q - \alpha r}{1 - \alpha}, r\right) \]
\[ = F(r, q') - F(r, q), \]

where the inequalities follow by Assumption 3, the first one by supermodularity of \( \Pi(p, r) \), and the second by its concavity in \( p \). Moreover, the constraint sets \( P_r \) are ascending with \( r \), i.e. for any \( r \leq r' \) and \( x \in P_r, x' \in P_{r'} \), it follows that \( \min(x, x') \in P_r \) and \( \max(x, x') \in P_{r'} \). Therefore, by Theorem 2.8.2 in Topkis (1998), \( q^*(r) \) is increasing in \( r \).

**F. Proof of Theorem 2**

Monotonicity of \( q^*(r) \) implies that the state path \( \{r_i^*\} \) is monotonic (by induction) in the bounded interval \( P \), hence it must converge to a steady state. The optimal prices \( p_i^* = \frac{r_i^* - \alpha r_i^*}{1 - \alpha} \) also converge to the same limit.

**G. Proof of Lemma 3**

In each case we show that \( W(p, r) \) is supermodular, which guarantees that the optimal policy \( p^*(r) \) is increasing in \( r \) (Topkis 1998, Theorem 2.8.2).

(a) If \( \alpha = 0 \) then \( W(p, r) = \Pi(p, r) + \beta V(p) \) is supermodular because \( \Pi(p, r) \) is so.

(b) Convexity of \( D(p, r) \) in \( r \) implies convexity of \( \Pi(p, r) \) in \( r \). It follows that the value function \( V(r) \) is convex in \( r \) (Smith and McCardle 2002, Proposition 5). Therefore, the objective function \( W(p, r) = \Pi(p, r) + \beta V(\alpha r + (1 - \alpha)p) \) is supermodular.

(c) Strong concavity of \( \Pi(r, p) \) insures that the value function is twice differentiable (Santos 1991). The envelope theorem for (2) gives \( V'(r) = W_r(p^*(r), r) \). Taking the second derivative with respect to \( r \), and using the implicit function theorem, we obtain

\[ V''(r) = W_{rr}(p^*(r), r) - \frac{W_{rr}^2(p^*(r), r)}{W_{pp}(p^*(r), r)}. \]  

(18)

The denominator above is negative by optimality of \( p^*(r) \), so \( V''(r) \geq W_{rr}(p^*(r), r) \). Writing \( W_{rr}(p, r) = \Pi_{rr}(p, r) + \alpha^2 \beta V''(\alpha r + (1 - \alpha)p) \), and using the assumption that \( \Pi_{rr} \geq -K \), we obtain

\[ V''(r) \geq \Pi_{rr}(p^*(r), r) + \alpha^2 \beta V''(\alpha r + (1 - \alpha)p) \]
\[ \geq \min_{r, p \in P} \{\Pi_{rr}(p, r) + \alpha^2 \beta V''(\alpha r + (1 - \alpha)p)\} \]
\[ \geq -K + \alpha^2 \beta \min_{r, p \in P} V''(\alpha r + (1 - \alpha)p). \]

Taking the minimum on the left hand side and rearranging terms, we obtain

\[ \min_{r \in P} V''(r) = \min_{r, p \in P} V''(\alpha r + (1 - \alpha)p) \geq \frac{-K}{1 - \alpha^2 \beta}. \]
This together with the assumption that $\Pi_{rp} \geq K/2$, implies

$$W_{pr}(p,r) = \Pi_{pr}(p,r) + \alpha(1 - \alpha)\beta V'(\alpha r + (1 - \alpha)p) \geq \frac{K}{2} - \delta K \geq 0,$$

where $\delta = \frac{\alpha(1 - \alpha)\beta}{1 - \alpha^2 \beta} \leq \frac{1}{2}$. Therefore $W(p,r)$ is supermodular.

### H. Proof of Proposition 2

Proposition 2 is a result of the following two lemmas.

**Lemma 6.**

$$(1 - \alpha^2 \beta)\Pi_{rp}(r,r) + (1 - \alpha)\alpha \beta \Pi_{rr}(r,r) \geq 0. \quad (19)$$

**Proof:** Define $\delta = \frac{(1 - \alpha)\alpha \beta}{1 - \alpha^2 \beta} \geq 0$. Condition (19) can be written as

$$\Pi_{rp}(p,p) + \delta \Pi_{rr}(p,p) \geq 0. \quad (20)$$

This holds if $\Pi_{rr}(p,p) \geq 0$, by supermodularity of $\Pi(p,r)$. On the other hand, if $\Pi_{rr}(p,p) < 0$ and because $\delta \leq 1/2$, it is sufficient to check that $\Pi_{rp}(p,p) + \frac{1}{2} \Pi_{rr}(p,p) \geq 0$. Because $R(0,r) = 0$ (Remark 1) we have $R_r(0,p) = R_{rr}(0,p) = 0$, and hence these terms vanish in:

$$\Pi_{rp}(p,p) = R_x(0,p) - p(R_{xx}(0,p) + R_{xr}(0,p)), \quad (21)$$

$$\Pi_{rr}(p,p) = p(R_{xx}(0,p) + 2R_{xr}(0,p)).$$

Combining these we obtain

$$\Pi_{rp}(p,p) + \frac{1}{2} \Pi_{rr}(p,p) = R_x(0,p) - \frac{1}{2} pR_{xx}(0,p) = -\frac{1}{2} \Pi_{pp}(p,p)^{\Delta} \geq 0, \quad (22)$$

by concavity of $\Pi^R(p,r)$ in $p$ (Assumption 3(c)). $\square$

**Lemma 7.** If Eq. (19) holds then the optimal pricing path is locally monotonic in the same direction as the reference price path.

**Proof:** Because $\Pi(r,p)$ is strongly concave and supermodular, the reference price policy $q^*(r)$ is $C^1$ (Araujo 1991). In fact, for our result we only need $q^*(r)$ to be continuously differentiable around the steady state. Other sets of sufficient (local) conditions, based on variations of strong concavity, are provided by Santos (1991) and Montrucchio (1998).

For local monotonicity, we want to show that $p^{**}(r^{**}) \geq 0$, i.e. $q^{**}(r^{**}) \geq \alpha$. Consider the Euler equation of problem (17) at any state $r$:

$$F_q(r,q^*(r)) + \beta F_r(q^*(r),q^*(q^*(r))) = 0. \quad (23)$$
Taking the derivative with respect to \( r \), we obtain that \( q''(r^{**}) \) solves:

\[
\frac{1 + \beta x^2}{x} = \eta, \quad \text{where} \quad \eta = -\frac{F_{qq}(r^{**}, r^{**}) + \beta F_{rr}(r^{**}, r^{**})}{F_{rr}(r^{**}, r^{**})}. \tag{24}
\]

The function on the left hand side is decreasing for \( x \leq 1 \). Therefore, either \( q''(r^{**}) > 1 \) or

\[
\frac{1 + \alpha^2 \beta}{\alpha} \geq \eta \tag{25}
\]

is equivalent to \( q''(r^{**}) \geq \alpha \).

If \( 1 < q^*(r) = \alpha r + (1 - \alpha)p^*(r) \), then \( p''(r^{**}) > 1 > 0 \), implying the desired result. Based on

\[
F(r, q) = \Pi_r \left( \frac{q - \alpha r}{1 - \alpha} \right),
\]

we calculate \( F_{qq} = \frac{1}{(1 - \alpha)^2} \Pi_{pp} \), \( F_{rr} = \Pi_{rr} + \frac{\alpha^2}{(1 - \alpha)^2} \Pi_{pp} - \frac{2\alpha}{1 - \alpha} \Pi_{rp} \), and

\[
F_{rq} = \frac{1}{1 - \alpha} \Pi_{rp} - \frac{\alpha}{(1 - \alpha)^2} \Pi_{pp}.
\]

Hence we can rewrite (25) as:

\[
(1 - \alpha^2 \beta) \Pi_{rp}(r^{**}, r^{**}) + (1 - \alpha)\alpha \Pi_{rr}(r^{**}, r^{**}) \geq 0,
\]

which is precisely condition (19) at the steady state. In particular, this proof shows that by negating (19) we can guarantee price monotonicity in the opposite direction. \( \square \)

I. Proof of Proposition 3

The proof is by contradiction. Assuming that an interior steady state \( r \) exists, and following the same line of proof as Theorem 1, in the loss seeking case Eq. (11) becomes

\[
(1 - \alpha \beta) \pi'(r) \geq r(\lambda_+(r) - \beta \lambda_-(r)). \tag{27}
\]

The right-hand side is derived from \( \lim_{\delta \to 0} \frac{R(\delta, r)}{\delta} = \lambda_+(r) \) and \( \lim_{\delta \to 0} \frac{R(-\delta, r - (1 - \alpha)\delta)}{\delta} = \lambda_-(r) - (1 - \alpha)R_e(0, r) = \lambda_-(r) \), because \( R_e(0, r) = 0 \) from \( R(0, r) = 0 \) (Remark 1).

The same argument for Eq. (12) gives

\[
(1 - \alpha \beta) \pi'(r) \leq r(\lambda_-(r) - \beta \lambda_+(r)). \tag{28}
\]

Combining (27) and (28), we obtain that if \( r \) is a steady state, then \( \lambda_-(r) - \beta \lambda_+(r) \geq \lambda_+(r) - \beta \lambda_-(r) \), that is, \( \lambda_-(r) \geq \lambda_+(r) \), contradicting \( (LS) \). Boundary steady states are excluded by the same argument as in Theorem 1.

J. Proof of Lemma 4

Because the reference effect \( R_\theta \) is a convex combination of the reference effects satisfying Assumptions 1-5, Problem \((P_\theta)\) satisfies these assumptions (all convex cone properties) as well.

For any initial state \( r \), by Eq. (10), the optimal policy of Problem \((K)\) is feasible for Problem \((P_\theta)\), and yields higher profit, i.e. \( V_\theta(r) \geq V_K(r) \). Let \( p_\theta^{**} \) be the steady state of Problem \((P_\theta)\). The constant pricing policy with initial state \( p_\theta^{**} \) is optimal for Problem \((P_\theta)\), feasible for Problem \((K)\) and yields the same value in both systems. Therefore, this constant pricing policy is also optimal for Problem \((K)\), and \( p_\theta^{**} \) is a steady state of Problem \((K)\).
K. Proof of Theorem 4

Consider the problem \((P_L)\) corresponding to the reference effect \(R_L(x, r)\) (i.e. \(\theta = 0\)). By observation, \(V_L(r) \geq V_K(r)\). Theorem 2 insures that when starting with a reference price below the steady state, \(r_0 < p_L^{*}\), the optimal state path for Problem \((P_L)\) monotonically increases to the steady state \(p_L^{*}\). This price path is feasible for Problem \((K)\). Because \(r_0 < p_L^{*}\), prices along this path exceed reference prices, so \(R_K(x, r) = R_L(x, r)\) on this path. Therefore \(V_L(r_0) = V_K(r_0)\), proving the first part.

The other case is proved similarly, by considering the problem \((P_G)\) corresponding to the reference effect \(R_G(x, r)\), and applying the result of Theorem 2.

From Lemma 4, the steady state \(p_0^*\) of Problem \((P_0)\) is also a steady state of Problem \((K)\). By Theorem 1, \(p_0^*\) solves \(1 - \alpha \beta \pi(p) - p(\theta \lambda_L(p) + (1 - \theta) \lambda_G(p)) = 0\). The function on the left hand side is continuously decreasing in \(p\) and continuous (linear) in \(\theta\). Therefore, as \(\theta \in [0, 1]\), the steady states \(p_0^*\) span the interval \([p_L^{*}, p_G^{*}]\). Hence this is the set of the steady states for Problem \((K)\).

L. Proof of Proposition 4

If \(p_L^{*} \leq r_0 \leq p_G^{*}\), Theorem 4 insures that the optimal price path is constant, so it is both globally and locally monotonic. If \(r_0 < p_L^{*}\), the proof of Theorem 4 shows that the optimal price path is the same as for Problem \((P_L)\). If \(R_L\) satisfies the conditions of Lemma 3 (respectively Proposition 2), the optimal price path is globally (respectively locally) monotonic, and so is the optimal price path for Problem \((K)\). The case \(r_0 > p_G^{*}\) is analogous.

M. Proof of Lemma 5

1. The last condition implies \(\Pi^R(p, r) = pR(r - p, r)\) is concave in \(p\), because

\[
\Pi_{pp}^R(p, r) = -2R_x(r - p, r) + pR_{xx}(r - p, r) \leq 0. \tag{29}
\]

Furthermore, \(R_x + R_r \geq 0\) by Remark 1, and \(R_{xx} + R_{xr} \leq 0\) by supermodularity of \(R(r - p, r)\) in \((p, r)\). This implies supermodularity of \(\Pi^R(p, r)\) because:

\[
\Pi_{pr}^R(p, r) = R_x(r - p, r) - pR_{xx}(r - p, r) + R_r(r - p, r) - pR_{xr}(r - p, r) \geq 0. \tag{30}
\]

2. If \(R\) is concave in \(x\) then (29) holds. Moreover, \(R_{xx} + R_{xr} \leq 0\) by \((DC)\), implying (30).

3. Eq. (29) holds if \(pR_{xx}(r - p, r) \leq R_x(r - p, r)\), so (a) insures concavity of \(\Pi^R(p, r)\) in \(p\). Together with \((DC)\), this implies Eq. (30) whenever \(R_x(r - p, r) \geq 0\). Otherwise, condition (b) implies \(R_r(r - p, r) - pR_{xr}(r - p, r) \geq 0\), which together with (a) gives Eq. (30). Hence \(\Pi^R(p, r)\) is supermodular.
N. Proof of Proposition 6

Under condition (b), consider the extended smooth reference effects: \( R_L(x,r) = x\lambda_-(r)1_{(x\geq 0)} + R_K(x,r)1_{(x\leq 0)} \) and \( R_G(x,r) = x\lambda_+(r)1_{(x\leq 0)} + R_K(x,r)1_{(x\geq 0)} \), as defined before the proposition. Condition (b) and convexity of \( R_K \) on the loss domain insure that \( R_K(x,r) = \min(R_G(x,r), R_L(x,r)) \), and \( \lambda_+(r) \geq \lambda_-(r) \) (LA). So Assumption 4 holds.

Assumption 3(a) is made, and 3(b) follows by condition (c). It remains to show that Assumption 3(c) holds for the problems with reference effects \( R_G \) and \( R_L \). It is easy to check that \( \Pi^R_{p}(p,r) \) is decreasing in \( p \), and increasing in \( r \). The result holds by Lemma 5 (part 2) for \( x \geq 0 \), i.e. \( p \leq r \). Otherwise, \( \Pi^R_{p}(p,r) = (r - 2p)\lambda_+(r) \) is decreasing in \( p \) and increasing in \( r \) because \( \lambda_+(r) > 0 \) is decreasing in \( r \) (by (DC)) and \( r\lambda_+(r) \) increasing in \( r \) (by condition (c)). Smoothness of \( R_G \) insures the global result. A similar argument can be made for \( \Pi^R_{p}(p,r) \), based on condition (a) of the proposition.

Under condition (b’), consider instead \( R_G(x,r) = R_K(x,r)/\rho(r)1_{(x\leq 0)} + R_K(x,r)1_{(x\geq 0)} \), where \( \rho(r) > 1 \) insures Eq.(5). It remains to verify that \( \Pi^R_{p}(p,r) = \frac{1}{\rho(r)}\Pi^R_{p}(p,r) \), for \( p > r \), is decreasing in \( p \) and increasing in \( r \). This follows from \( \rho(r) \) increasing in \( r \) and from \( \Pi^R_{p}(p,r) \) negative, decreasing in \( p \) and increasing in \( r \) on \( p > r \) by part (a).

Endnotes

1. We provide some intuition for a transient high-low pricing strategy, such as the one depicted in Figure 4. The firms’ pricing decision in each period affects three components of demand, and their associated profits: reference profit \( \Pi^R(p,r) \), base profit \( \pi(p) \) (without reference effects), and future profit \( V(\alpha r + (1 - \alpha)p) \). Starting with a low reference price, the firm initially charges a relatively high price to capitalize on base and future profit. This creates a strong negative market perception (and reference profit), which the firm tries to assuage by temporarily decreasing prices. In this transient stage, the firm focuses on limiting losses from the reference effect. Some base and future profit benefits are foregone (e.g. relative to a constant pricing policy), but the latter continues to increase, as reference prices increase. Once the negative reference effect has been sufficiently diminished (by closing the gap between price and reference price), the firm can again capitalize on base and future profits, by increasing prices (while keeping the gap between price and reference price small).

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References


